MATHEMATICS

Paper – II

*Time Allowed : **Three** Hours*

*Maximum Marks : **200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

*Answers must be written in **ENGLISH** only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. *8*

(b) Show that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$. *8*

(c) Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when x is rational} \\ 1 & \text{when x is irrational} \end{cases}$$

on the interval [0, 1]. *8*

(d) Using Cauchy's Integral formula, evaluate the integral $\oint_c \frac{dz}{(z^2 + 4)^2}$ where $c : |z - i| = 2$. *8*

(e) A firm manufactures two products A and B on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M1 and M2. Product A requires one minute of processing time on M1 and two minutes on M2, while B requires one minute on M1 and one minute on M2. Machine M1 is available for not more than 7 hours 30 minutes, while machine M2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactured to get maximum profit, using graphical method. *8*

Q2. (a) Let I and J be ideals in a ring R. Then prove that the quotient ring $(I + J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. *10*

(b) Show that the integral $\int_0^{\pi/2} \log \sin x \, dx$ is convergent and hence evaluate it. *15*

(c) If f(z) is analytic in a domain D and $|f(z)|$ is a non-zero constant in D, then show that f(z) is constant in D. *15*

Q3. (a) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find the order of b. *10*

(b) Show that the sequence $\{\tan^{-1} nx\}$, $x \geq 0$ is uniformly convergent on any interval [a, b], a > 0 but is only pointwise convergent on [0, b]. *15*

(c) Use simplex method to solve the following problem : *15*

Maximize $z = 2x_1 + 5x_2$

subject to $x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$

Q4. (a) Show that the smallest subgroup V of A_4 containing (1, 2) (3, 4), (1, 3) (2, 4) and (1, 4) (2, 3) is isomorphic to the Klein 4-group. *10*

(b) Classify the singular point z = 0 of the function $f(z) = \dfrac{e^z}{z + \sin z}$ and obtain the principal part of the Laurent series expansion of f(z). *15*

(c) A salesman wants to visit cities C1, C2, C3 and C4. He does not want to visit any city twice before completing the tour of all the cities and wishes to return to his home city, the starting station. Cost of going from one city to another in rupees is given below in the table. Find the least cost route. *15*

From City \ To City	C1	C2	C3	C4
C1	0	30	80	50
C2	40	0	140	30
C3	40	50	0	20
C4	70	80	130	0

SECTION B

Q5. (a) Find the solution of the equation :

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y.$$ 8

(b) The following table gives the values of $y = f(x)$ for certain equidistant values of x. Find the value of f(x) when $x = 0{\cdot}612$ using Newton's forward difference interpolation formula. 8

x :	0·61	0·62	0·63	0·64	0·65
y = f(x) :	1·840431	1·858928	1·877610	1·896481	1·915541

(c) Following values of x_i and the corresponding values of y_i are given. Find $\int_0^3 y\,dx$ using Simpson's one-third rule. 8

x_i :	0·0	0·5	1·0	1·5	2·0	2·5	3·0
y_i :	0·0	0·75	1·0	0·75	0·0	– 1·25	– 3·0

(d) Consider the flow field given by $\psi = a(x^2 - y^2)$, 'a' being a constant. Show that the flow is irrotational. Determine the velocity potential for this flow and show that the streamlines and equivelocity potential curves are orthogonal. 8

(e) Find a complete integral of the equation by Charpit's method $p^2x + q^2y = z$. Here $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. 8

Q6. (a) Test the integrability of the equation

$z(z + y^2)\,dx + z(z + x^2)\,dy - xy(x + y)\,dz = 0.$

If integrable, then find its solution. 15

(b) Solve the following system of equations by Gauss-Jordan elimination method : *10*

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + x_3 = 6$$

$$x_1 - x_2 - x_3 = -3$$

(c) For a dynamical system

$$T = \frac{1}{2}\{(1 + 2k)\,\dot{\theta}^2 + 2\dot{\theta}\dot{\varphi} + \dot{\varphi}^2\},$$

$$V = \frac{n^2}{2}\{(1 + k)\,\theta^2 + \varphi^2\},$$

where θ, φ are coordinates and n, k are positive constants, write down the Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\varphi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \varphi) = 0.$$

Further show that if $\theta = \varphi$, $\dot{\theta} = \dot{\varphi}$ at $t = 0$, then $\theta = \varphi$ for all t. *15*

Q7. (a) Given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Find $y(0{\cdot}1)$ and $y(0{\cdot}2)$ by fourth order Runge-Kutta method. *15*

(b) Consider a mass-spring system consisting of a mass m and a linear spring of stiffness k hanging from a fixed point. Find the equation of motion using the Hamiltonian method, assuming that the displacement x is measured from the unstretched position of the string. *10*

(c) Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy = z + c$. *15*

Q8. (a) Consider that the region $0 \leq z \leq h$ between the planes $z = 0$ and $z = h$ is filled with viscous incompressible fluid. The plane $z = 0$ is held at rest and the plane $z = h$ moves with constant velocity $V\hat{j}$. When conditions are steady, assuming there is no slip between the fluid and either boundary, and neglecting body forces, show that the velocity profile between the plates is parabolic. Find the tangential stress at any point P(x, y, z) of the fluid and determine the drag per unit area on both the planes. *15*

(b) State the Newton–Raphson iteration formula to compute a root of an equation f(x) = 0 and hence write a program in BASIC to compute a root of the equation

$$\cos x - xe^x = 0$$

lying between 0 and 1. Use DEF function to define f(x) and f′(x). *10*

(c) Use Gauss quadrature formula of point six to evaluate $\int_0^1 \frac{dx}{1+x^2}$ given

$x_1 = -0{\cdot}23861919, \quad w_1 = 0{\cdot}46791393$

$x_2 = -0{\cdot}66120939, \quad w_2 = 0{\cdot}36076157$

$x_3 = -0{\cdot}93246951, \quad w_3 = 0{\cdot}17132449$

$x_4 = -x_1,\ x_5 = -x_2,\ x_6 = -x_3,\ w_4 = w_1,\ w_5 = w_2$ and $w_6 = w_3$. *15*